Exotic smoothness, noncommutative geometry and particle physics.

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Abstract

We investigate how exotic differential structures may reveal themselves in particle physics. The analysis is based on the A. Connes' construction of the standard model. It is shown that, if one of the copies of the spacetime manifold is equipped with an exotic differential structure, compact object of geometric origin may exist even if the spacetime is topologically trivial. Possible implications are discussed. An $SU(3) \otimes SU(2) \otimes U(1)$ gauge model is constructed. This model may not be realistic but it shows what kind of physical phenomena might be expected due to the existence of exotic differential structures on the spacetime manifold.

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There is no interesting topology on $\mathbf{R^4}$, the Euclidian four-dimensional space (or to be more precise it is topologically equivalent to a single point space). The counter-intuitive results [1–5] that $\mathbf{R^4}$ may be given infinitely many exotic differential structures raised question of their physical consequences [6–7]. An exotic differential structure $\hat{C}^k(M)$ on a manifold M is, by definition, a differential structure that is not diffeomorphic to the one considered as a standard one, $C^k(M)$. This means that the sets od differentiable functions are different. For example, there are functions on $\mathbf{R^4}$ that are not differentiable on some exotic $\mathbf{R^4_\Theta}$ which is homeomorphic but not diffeomorphic to $\mathbf{R^4}$. Here we would like to investigate the role that exotic differential structures on the spacetime manifold may play in particle physics. Our starting point will be the A. Connes' noncommutative geometry based construction of the standard model [8–15]. A. Connes managed to reformulate the standard notions of differential geometry in a pure algebraic way that allows to get rid of the differentiability and continuity requirements. The notion of spacetime manifold S can be equivalently described by the (commutative) algebra $C^\infty(S)$ of smooth functions on S and can be generalized to (a priori) an arbitrary noncommutative algebra. Fiber bundles became projective modules in this language. A properly generalized connection can describe gauge fields on these objects. This allows to incorporate the Higgs field into the gauge field so that the correct (that is leading to spontaneously broken gauge symmetry) form of the scalar potential is obtained. The reader

is referred to [8–15] for details.

We shall consider the algebra A:

$$A = M_1 \left(C^{\infty} \left(S \right) \right) \oplus M_2 \left(C^{\infty} \left(S \right) \right) \oplus M_1 \left(\hat{C}^{\infty} \left(S \right) \right) \oplus M_3 \left(\hat{C}^{\infty} \left(S \right) \right), \qquad (1)$$

where $M_i(ring)$ denotes $i \times i$ matrices over the ring $C^{\infty}(S)$ or $\hat{C}^{\infty}(S)$. The hat denotes that the functions are smooth with respect to some nonstandard differential structure on S. The free Dirac operator has the form:

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes Id & \gamma_5 \otimes m_{12} & \gamma_5 \otimes m_{13} & \gamma_5 \otimes m_{14} \\ \gamma_5 \otimes m_{21} & \partial\!\!\!/ \otimes Id & \gamma_5 \otimes m_{23} & \gamma_5 \otimes m_{24} \\ \gamma_5 \otimes m_{31} & \gamma_5 \otimes m_{32} & \hat{\partial\!\!\!/} \otimes Id & \gamma_5 \otimes m_{34} \\ \gamma_5 \otimes m_{41} & \gamma_5 \otimes m_{42} & \gamma_5 \otimes m_{43} & \hat{\partial\!\!\!/} \otimes Id \end{pmatrix}, \qquad (2)$$

here, as before, the hat denotes the "exoticness" of the appropriate differential structure. The parameters m_{ij} describe the fermionic mass sector. Let ρ be a (self-adjoint) one-form in $\Omega^1(A) \subset \Omega^*(A)$, here $\Omega^*(A)$ denotes the universal differential algebra of A [8,9]:

$$\rho = \sum_i a_i db_i \ , \quad a_i, b_i \varepsilon A. \qquad (3)$$

We will use the following notation for an $a \varepsilon A$

$$a = diag \left(a^1, a^2, a^3, a^4 \right) \qquad (4)$$

with a^i belonging to the appropriate matrix algebra in (1). The physical bosonic fields are defined via the representation π in terms of (bounded) operators in the appropriate Hilbert space [6–12]:

$$\pi\left(a_0 da_1 \dots a_n\right) = a_0 \left[D, a_1\right] \dots \left[D, a_n\right]. \tag{5}$$

Standard calculations lead to

$$\pi\left(\rho\right) = \begin{pmatrix} A^1 & \gamma_5 \otimes \phi^{12} & \gamma_5 \otimes \phi^{13} & \gamma_5 \otimes \phi^{14} \\ \gamma_5 \otimes \phi^{21} & A^2 & \gamma_5 \otimes \phi^{23} & \gamma_5 \otimes \phi^{24} \\ \gamma_5 \otimes \phi^{31} & \gamma_5 \otimes \phi^{32} & A^3 & \gamma_5 \otimes \phi^{34} \\ \gamma_5 \otimes \phi^{41} & \gamma_5 \otimes \phi^{42} & \gamma_5 \otimes \phi^{43} & A^4 \end{pmatrix}, \tag{6}$$

where

$$A^p = \sum_i a_i^p \not\partial b_i^p \ , \qquad p = 1, 2 \tag{7a}$$

$$A^p = \sum_i a_i^p \hat{\not\partial} b_i^p \ , \qquad p = 3, 4 \tag{7b}$$

and

$$\phi^{pq} = \sum_i a_i^p \left(m_{pq} b_i^q - b_i^p m_{pq}\right) \quad p \neq q. \tag{8}$$

Note, that the A^3 and A^4 are given in terms of the exotic differential structure. They will be the source of the $SU(3)$ part of the gauge group. The additional $U(1)$ term A^3 is the price we have to pay for the "exactness" of the $SU(3)$ gauge symmetry: noncommutative geometry prefers broken gauge symmetries. It is still an open question if noncommutative geometry provides us with new unbroken symmetries, see Ref. 8-11 for details. There is one subtle step in the reduction of the gauge symmetry from $SU(2) \otimes U(1) \otimes U(1) \otimes SU(3)$ to $SU(2) \otimes U(1) \otimes SU(3)$. Namely, one should require that the $U(1)$

part of the associated connection is equal to Y, the $U(1)$ part of the $SU(3)$ connection and the "exotic" $U(1)$ factor is equal to $-Y$. A more elegant but equivalent treatment can be found Ref. 9. But these are defined with respect to different differential structures! This can be done only locally as the exotic differential structure defines different set of smooth function than the standard one (and vice versa). We will return to this problem later. This defines the algebraic structure of the standard model. To obtain the Lagrangian, we have to calculate the curvature Θ, $\Theta = \pi(d\rho) = \sum_i [D, a_i][D, b_i]$. This can be easily done. The bosonic part of the action is given by the formula

$$I_{YM} = Tr_\omega \left(\Theta^2 |D|^{-4} \right) \ , \tag{9}$$

where Tr_ω is the Diximier trace defined by [8,9]

$$Tr_\omega \left(|O| \right) = \lim \frac{1}{logN} \sum_{i=0}^{i=N} \mu_i \left(O \right) \ . \tag{10}$$

Here μ_i denotes the $i-$th eigenvalue of the (compact) operator O. The Diximier trace gives the logarithmic divergencies, and gives zero for operators in the ordinary trace class. We will use the heat kernel method [16−20]. For a second order positive pseudodifferential operator $O\ : L^2(E) -> \ L^2(E)$, where $L^2(E)$ denotes the space square integrable functions on the vector bundle E, the operator

$$e^{-tO} = \frac{1}{2\pi i} \int_C e^{-t\zeta} \left(\zeta Id - O \right)^{-1} d\zeta \tag{11}$$

is well defined for $Re\ t > 0$ [16–18]. Then the Mellin transformation [16]

$$\int_0^\infty e^{-tO} t^{s-1} dt = \Gamma(s)\, O^{-s} \tag{12}$$

provides us with the formula:

$$|D|^{-4} = \int_0^\infty dt\ t e^{-t|D|^2}\ . \tag{13}$$

Now, we have to restrict ourselves to the case $m_{31} = m_{32} = m_{41} = m_{42} = m_{13} = m_{14} = m_{23} = m_{24}$ in (2) so that the free Dirac operator takes the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\ , \tag{14}$$

where D_2 is defined with respect to an exotic differential structure. This allows us to calculate the Diximier trace and the notion of a point retains its ordinary spacetime sense. This is not very restrictive as the $SU(3)$ gauge symmetry is unbroken. Calculation of the Diximier trace in the general case is more involved (if possible) and we would loose the convenient spacetime interpretation. The formula [18]

$$e^{-t(D_1 \oplus D_2)} = e^{-t(D_1)} \oplus e^{-t(D_2)} \tag{15}$$

leads to the following asymptotic formula:

$$tr\left(\left(f\oplus\hat{f}\right)e^{-t|D|^2}\right)=\int dx^4\sqrt{g}f\left(\frac{a_0}{t^2}+\frac{a_1}{t}+\ldots\right)+\int\hat{d}x^4\sqrt{\hat{g}}\hat{f}\left(\frac{\hat{a}_0}{t^2}+\frac{\hat{a}_1}{t}+\ldots\right), \tag{16}$$

where a_i are the spectral coefficients [16–20], g is the metric tensor, dots denote the finite terms in the limit $t->0$ and the hat distinguishes between the standard and exotic structures. For the Dirac Laplace'ans $|D_i|^2$ $i=1,2$ we have $a_1=1$ and a_2 is equal to the curvature R. This gives the the following value of the Yang-Mills (bosonic) action (roughly speaking this is the "logarithmic divergence" term):

$$I_{YM}=\frac{1}{4}\int dx^4\sqrt{g}\,TR\left(\pi^2\left(\theta\right)\right)+\int\hat{d}x^4\sqrt{\hat{g}}\,TR\left(\pi^2\left(\hat{\theta}\right)\right)\ , \tag{17}$$

where the the trace TR is taken over the Clifford algebra and the matrix structure. As before, the hat is used to distinguish the "exotic" part of the curvature from the "non-exotic" one. Note, that due to continuity, the two integrals do not feel the different differential structures, so formally, the action looks the same as in the ordinary case. Now, standard algebraic calculations (after elimination of spurious degrees of freedom by hand [8,10,12–15] or by going to the quotient space [9]) lead to the following Lagrangian (in the Minkowski space):

$$\begin{aligned} L_{YM}= &\int\sqrt{g}\{\tfrac{1}{4}N_g\left(F^1_{\mu\nu}F^{1\mu\nu}+F^2_{\mu\nu}F^{2\mu\nu}+F^c_{\mu\nu}F^{c\mu\nu}\right)\\ &+\tfrac{1}{2}Tr\left(mm^\dagger\right)|\partial\phi+A_1\phi-\phi^\dagger A_2|^2\\ &-\tfrac{1}{2}\left(Tr\left(mm^\dagger\right)^2-\left(Trmm^\dagger\right)^2\right)\left(\phi\phi^\dagger-1\right)^2\}d^4x\ . \end{aligned} \tag{18}$$

The $SU(3)_c$ stress tensor $F^c_{\mu\nu}F^{c\mu\nu}$ is defined with respect to the exotic differential structure. We will not need the concrete values of the traces in (18) so will not quote them (they are analogous to those in [21–22]). Fermion fields are added in the usual way [8–15]:

$$\begin{aligned} L_f &= <\psi|D+\pi\left(\rho\right)|\psi> \\ &= \int\left(\bar{\psi}_L \not{D}\psi_L + \bar{\psi}_R \not{D}\psi_R + \bar{\psi}_L\phi\otimes m\psi_R + \bar{\psi}_R\phi^\dagger\otimes m^\dagger\psi_L\right) d^4x \ , \end{aligned} \tag{19}$$

where we have included the $\pi(\rho)$ term into $\not{D}$. The quark fields are defined with respect to the exotic differential structure. To proceed, let us review some results concerning exotic differential structures on $\mathbf{R^4}$ [5–7].

An exotic $\mathbf{R^4_\Theta}$ consists of a set of points which can be globally continuously identified with the set four coordinates (x^1, x^2, x^3, x^4). These coordinates may be smooth locally but they cannot be globally continued as smooth functions and no diffeomorphic image of an exotic $\mathbf{R^4_\Theta}$ can be given such global coordinates in a smooth way. There are uncountable many of different $\mathbf{R^4_\Theta}$. C. H. Brans has proved the following theorem [7]:

Theorem 1. *There exist smooth manifolds which are homeomorphic but not diffeomorphic to* $\mathbf{R^4}$ *and for which the global coordinates* (t, x, y, z) *are smooth for* $x^2 + y^2 + z^2 \geq a^2 > 0$*, but not globally. Smooth metrics exist for which the boundary of this region is timelike, so that the exoticness is spatially confined.*

He has also conjectured that such localized exoticness can act as an source for some externally regular field, just as matter or a wormhole can. Of course, there are also $\mathbf{R^4_\Theta}$ whose exoticness cannot be localized. They might have important cosmological consequences. We also have [7]

Theorem 2. *If M is a smooth connected 4-manifolds and S is a closed submanifold for which $H^4(M, S, \mathbf{Z}) = 0$, then any smooth, time-orientable Lorentz metric defined over S can be smoothly continued to all of M.*

Now we are prepared to analyse the Lagrangian given by (18). Despite the fact that it looks like an ordinary one we should remember that the strongly interacting fields are defined with respect to an exotic differential structure. This means that, in general, these fields may not be smooth with respect to the standard differential structure, although they are smooth solutions with respect to the exotic one. They certainly are continuous. In general, only those "exotic" fields that vanish outside a compact set (not necessary containing the exotic region) can be expected to be differentiable with respect to the standard differential structure (this is because manifolds are locally Euclidean and constant functions are differentiable) and consistent with the derivation of the Lagrangian (18). Theorem 2. suggests that it might be possible to continue a Lorentz structure to all of spacetime so that (18) make sense (e.g. for a non-compact manifold M, submanifolds S for which $H^3(S; \mathbf{Z}) = 0$ fulfil the required conditions [7]). This means that strongly in-

teracting fields probably must vanish outside a compact set to be consistent with the standard (?) differential structure that governs electroweak sector. One can say that the exotic geometry confines strongly interacting particles to live inside bag-like structures. We do not claim (although it might be so) that we have found a solution to the confinement problem, but these results are really astonishing. Unfortunatelly, the estimation of the size of such an object is not possible without at the moment not available information on the global structure of exotic manifolds. A priori, they may be as small as baryon or as big as quark star. What important is is the fact that such object are not black-hole-like ones. It is possible to "get inside such an object and go back". There is no topological obstruction that can prevent us entering the exotic region: everything is smooth but some fields must have compact supports. One may investigate its structure as one does in the case of baryons via electroweak interactions. Of course, the above analysis is classical: we do not know how to quantize models that noncommutative geometry provides us with. Let us conclude by saying that exotic differential structures over spacetime may play important role in particle physics. They may provide us with "confining forces" of pure geometrical origin: one do not have to introduce additional scalar fields to obtain bag-like models. We have discussed only exotic versions of $\mathbf{R}^4$ but there are also other exotic 4-manifolds. The proposed model is probably far from being a realistic one but it is the only one ever constructed. We have conncted the geometrical

exoticness with strong interactions. We can give only one reason for doing so. A. Conne's construction provides us with spontaneously broken gauge symmetries. Exact gauge symmetries are "out of the way" so we have made the $SU(3)_{color}$ sector "spatially-exotic". One can also ask the question *if one writes a Lagrangian, must all of its terms be defined with respect to the same differential structure on spacetime?* The answer is not so obvious. Obviously, the topic deserves further investigation. One of the most important questions is *how do exotic differential structures influence quantum theory?* This is under investigation.

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